On Extension of Twisted Canonical forms

(joint with J. Cao)

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- We have a projection map $\pi_k : \mathcal{F}_{k+1} \to \mathcal{F}_k$

Question

Let $s \in H^0(\mathcal{X}, \mathcal{F}_k)$. When does s belongs to the image of π_k?

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$$(\star) \qquad \int_X \left| \frac{\Lambda_k}{dt} \right|^2 e^{-(1-\varepsilon)\varphi_L} dV < \infty, \qquad \forall \varepsilon > 0.$$

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• For a more general result –involving an *abstract* $\mathcal{L}$– we refer to arXiv:2012.05063.

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• We discuss next the proof of Theorem [CP].

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- The first step is to show the following.

Claim

There exist α and β forms of type $(n, 0)$ and $(n-1, 1)$ with values in L respectively, such that their coefficients are $\frac{\mathcal{C}^\infty}{s^N}$ and

$$\lambda_k = \overline{\partial}\alpha + D'\beta$$

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    - Let $\mathrm{Lie}_\Xi := D'(\Xi \rfloor \cdot)$ so that we have
    $$\mathrm{Lie}_\Xi : \mathcal{C}^\infty_{n+1,q}(\mathcal{X}, \mathcal{L}) \to \mathcal{C}^\infty_{n+1,q}(\mathcal{X}, \mathcal{L})$$

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- Summing up, after $k+1$ derivatives we get

$$\overline{\partial}\alpha + D'\beta = \lambda_k$$

on $X \setminus (s=0)$.

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  - The inverse image λ_k induces a $(n, 1)$ form $\widehat{\lambda}_k$ with values in $E + L$ where $L \equiv \sum \delta_i F_i$ for $\delta_i \in]0, 1[\cap \mathbb{Q}$.

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  - Let h_L be the metric on L given by $\varphi_L := \sum \delta_i \log |f_i|^2$.
- We have $\overline{\partial}\widehat{\alpha} + D'\widehat{\beta} = \widehat{\lambda}_k$ on $\widehat{X} \setminus (E \cup F)$, forms with values in $E + L$.

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  - $V_1 := \sum_{i\in I} \theta_i z_i^1 \frac{\partial}{\partial z_i^1}$; consider $\alpha_1 := \alpha + \frac{1}{N} D'\left(V_1 \rfloor \alpha\right)$ (here D' is the Chern connection on $E + L$ and N is the pole order of α along E_1.

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  - The pole order of α_1 is smaller than $N - 1$.

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There exist α and β forms of type $(n,0)$ and $(n-1,1)$ with values in L and logarithmic poles along E such that

$$\overline{\partial}\alpha + D'\beta = \frac{\lambda_k}{s_E} \qquad \text{on } X \setminus E.$$

- Assume that $E = E_1$ and $F = 0$; we argue as follows
  - $\big(\Omega_i, (z_i)\big)_{i\in I}$ finite covering of X, such that $E_1 \cap \Omega_i = (z_i^1 = 0)$.
  - $V_1 := \sum_{i\in I} \theta_i z_i^1 \frac{\partial}{\partial z_i^1}$; consider $\alpha_1 := \alpha + \frac{1}{N} D'\left(V_1 \rfloor \alpha\right)$ (here D' is the Chern connection on $E + L$ and N is the pole order of α along E_1.
  - The pole order of α_1 is smaller than $N-1$.
  - We do similar manipulations with β; the relevant facts are $[\overline{\partial}, D'] = 0$ on $X \setminus (E \cup F)$ and $D' \circ D' = 0$.

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Let λ be a smooth $\overline{\partial}$-closed $(n,1)$-form on X with values in $E+L$ such that.

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Then λ is $\overline{\partial}$-exact.

- Main tool: Hodge decomposition, version that we next discuss.

Hodge decomposition

- Consider a compact Kähler manifold X and a snc divisor $Y = Y_1 + \ldots Y_k$.

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$$\omega_{\mathcal{P}}|_{\Omega} = \sum_{i=1}^{r} \frac{\sqrt{-1}dz_i \wedge d\overline{z}_i}{|z_i|^2 \log^2 |z_i|^2} + \sum_{i=r+1}^{n} \sqrt{-1}dz_i \wedge d\overline{z}_i.$$

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Theorem 3

We have the following decomposition for $(X, \omega_{\mathcal{P}})$ and (L, h_L).

$$L^2_{n,1}(X_0, L) = \mathcal{H}_{n,1}(X_0, L) \oplus \mathrm{Im}\overline{\partial} \oplus \mathrm{Im}\overline{\partial}^{\star}$$

where $X_0 := X \setminus Y$.

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Let $p \leq n$ be an integer. There exists a positive constant $C > 0$ such that

$$\int_{X_0} |u|^2_{\omega_{\mathcal{P}}} e^{-\varphi_L} dV \leq C \int_{X_0} |\overline{\partial} u|^2_{\omega_{\mathcal{P}}} e^{-\varphi_L} dV_{\omega_{\mathcal{P}}}$$

for any L-valued form u of type $(p, 0)$ which belongs to the domain of $\overline{\partial}$ and which is orthogonal to the space of L^2 harmonic $(p, 0)$–forms.

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- Application: same results hold for metrics with conic singularities along Y

$$\omega_{\mathcal{C}}|_{\Omega} = \sum_{i=1}^{r} \frac{\sqrt{-1} dz_i \wedge d\overline{z}_i}{|z_i|^{2\frac{m_i - 1}{m_i}}} + \sum_{i=r+1}^{n} \sqrt{-1} dz_i \wedge d\overline{z}_i.$$

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- Consider the complete metric $\omega_{\mathcal{C},\epsilon} = \omega_{\mathcal{C}} + \epsilon\omega_{\mathcal{P}}$
- Use Theorem 4 and $\epsilon \to 0$
- Main point is that the space of L^2 holomorphic p forms is *independent* of ϵ.

The klt case

- Back to the equation $\dfrac{\lambda}{s_E} = \overline{\partial}\alpha + D'\beta$. If $E = 0$, we argue as follows.

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  - We use the fact that $\gamma_\xi := \star\xi$ is holomorphic.

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- Let $(\theta_\epsilon)_{\epsilon>0}$ truncation functions corresponding to $E+F$; we have

$$\int_X \frac{\lambda}{s_E} \wedge \overline{\gamma}_\xi e^{-\varphi_L} = \lim_{\epsilon\to 0} \int_X \theta_\epsilon \frac{\lambda}{s_E} \wedge \overline{\gamma}_\xi e^{-\varphi_L}$$

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- We have $\displaystyle\lim_{\epsilon\to 0} \int_X \theta_\epsilon D'\beta \wedge \overline{\gamma}_\xi e^{-\varphi_L} = 0$ (even if now β has logarithmic poles).

End of the proof

- The other term is more troublesome:

$$\lim_{\epsilon \to 0} \int_X \theta_\epsilon \overline{\partial} \alpha \wedge \overline{\gamma}_\xi e^{-\varphi_L} = \sum_i \int_{E_i} \alpha_i \wedge \overline{\gamma}_\xi e^{-\varphi_L}$$

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- α_i is a $(n-1,0)$ form on E_i with logarithmic poles on $E - E_i|_{E_i}$. By interpreting it as current on E_i we get

$$\alpha_i = \tau_i + \Delta''(\mathcal{G}_i)$$

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- $\Delta''(\mathcal{G}_i)$ is orthogonal on harmonic forms, so

$$\int_{E_i} \alpha_i \wedge \overline{\gamma}_\xi e^{-\varphi_L} = \int_{E_i} \tau_i \wedge \overline{\gamma}_\xi e^{-\varphi_L}$$

End of the proof

- In conclusion, the current

$$\phi \to \int_X \frac{\lambda}{s_E} \wedge \overline{\phi} e^{-\varphi_L} + \sum_i \int_{E_i} \tau_i \wedge \overline{\phi} e^{-\varphi_L}$$

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- It follows that it is $\overline{\partial}$-exact; multiplication s_E shows that λ is $\overline{\partial}$-exact.